Exercise 34

Use the Intermediate Value Theorem to show that there is a root of the equation in the given interval.

$$\cos\sqrt{x} = e^x - 2, \quad (0, 1)$$

Solution

Bring all terms to one side.

$$\cos\sqrt{x} - e^x + 2 = 0$$

Let $f(x) = \cos\sqrt{x} - e^x + 2$. The cosine, exponential, and constant functions are continuous at all numbers in the same domain, $(-\infty, \infty)$. The square root function is also continuous but on its own restricted domain, $[0, \infty)$. Consequently, $f(x)$ is continuous on $[0, \infty)$, which includes the closed interval $[0, 1]$. Find a value of x in this closed interval for which $f(x)$ is negative, and find a value of x in this closed interval for which $f(x)$ is positive.

$$f(0) = \cos\sqrt{0} - e^0 + 2 \approx 2$$

$$f(1) = \cos\sqrt{1} - e^1 + 2 \approx -0.180$$

$N = 0$ lies between $f(0)$ and $f(1)$, so by the Intermediate Value Theorem, there exists a root in the open interval $(0, 1)$.